ON SIMPLE GROUPS WITH A CYCLIC MAXIMAL 2-SYLOW INTERSECTION

BY

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ABSTRACT

Finite simple groups G with a cyclic maximal 2–Sylow intersection V are classified under the assumption that $[G: N_G(V)]$ is odd.

1. Introduction

This paper is a step toward classification of simple groups with a cyclic maximal 2-Sylow intersection. We prove the following:

THEOREM. *Let G be a nonabelian finite simple group. Suppose that V is a maximal 2-Sylow intersection of G satisfying the following conditions:*

i) *V is cyclic, and*

ii) *$|G:N_G(V)|$ is odd.*

Then G is isomorphic to one of the following groups:

A) $PSL(2,q)$, $q \neq 2^n \pm 1$ *for some* $n > 2$ *and* $q \neq 3$.

B) $PSL(3,q)$, $q \equiv -1 \bmod 4)$, *but* $q \neq 2^n - 1$ *for some* $n \geqq 2$.

C) $PSU(3,q)$, $q \equiv 0$ or $1 (\bmod 4)$, *but* $q \neq 2^n + 1$ *for some* $n \geqq 2$.

D) $S_Z(q)$.

E) J_{11}.

Conversly, all groups mentioned satisfy the assumptions of the theorem.

In Section 2, groups satisfying the conditions of the Theorem will be classified under the assumption that $N_G(V)$ is solvable. The nonsolvable case will be dealt with in Section 3.

Our notation is standard. If H is a group, $K \subset H$ means that K is a proper

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subgroup of H and $O(H)$ denotes the maximal normal sugbroup of H of odd order. We will say that H is a TI-group if the intersection of two distinct S_2-subgroups of H is always 1. The groups $PSL(2,2^n)$, $n \geqq 2$, $PSU(3,2^n)$, $n \geqq 2$ and $S_Z(2^n)$, $n \geqq 3$ will be called *Bender simple groups*. The simple group of Janko of order 175,560 will be denoted by J_{11}. Finally, if H is a group of permutations on a set Ω and Δ is a subset of Ω, the subgroup of H fixing the elements of Δ will be denoted by H_Δ.

2. Solvable $N_G(V)$

We will prove the following:

PROPOSITION 1. *Under the assumptions of the Theorem, suppose that $N_G(V)$ is solvable. Then G is isomorphic to one of the following groups:*

I) $PSL(3,q)$, $q \equiv -1 \pmod 4$, *but* $q \neq 2^{n-2}-1$ *for some* $n \geqq 4$.

II) $PSU(3,q)$, $q \equiv 1 \pmod 4$, *but* $q \neq 2^{n-2}+1$ *for some* $n \geqq 4$.

III) $PSL(2,q)$, *q odd, but* $q \neq 2^n \pm 1$ *for some* $n > 2$ *and* $q \neq 3$. *Conversely, all groups mentioned satisfy the assumptions of the proposition.*

PROOF. Let $H \equiv N_G(V)$; since H is solvable, $V \neq 1$. The maximality of V forces H/V to be a solvable non-2-closed TI-group. Thus, by [8], an S_2-subgroup of H/V is of 2-rank 1, hence in view of assumptions (i)–(ii), 2-rank $G = 2$. It follows then by [2] that an S_2-subgroup R of H (hence of G) is of one of the following types: (a) dihedral, (b) quasi-dihedral, (c) wreathed, or (d) isomorphic to S_2-subgroup of $PSU(3,4)$. Moreover, it follows by [3] that if R/V is cyclic, then $|R:V| = 2$.

Cases (d) and (c) are impossible, since then R does not possess a normal cyclic subgroup V such that R/V is cyclic of order 2 or generalized quaternion.

Suppose now that R is quasi-dihedral of order 2^n (Case b). Then R/V is cyclic of order 2 and by [1], one of the following holds: (i) $G \simeq M_{11}$, (ii) $G \simeq PSL(3,q)$, $q \equiv 1 \pmod 4$, (iii) $G \simeq PSU(3,q)$, $q \equiv 1 \pmod 4$. Clearly V is a 2-Sylow intersection if and only if H is not 2-closed. In case (i), $C_G(V) = V$ hence H is a 2-group, contrary to our assumptions. In case (ii), let z be the involution of V. Then by [1], $C_G(z) \simeq GL(2,q)/D$, where D is a central subgroup of $GL(2,q)$ of odd order. Let $K \equiv GL(2,q)$ and let V_1 be a subgroup of K such that $V \simeq V_1D/D$ and $V_1 \cap D = 1$. As $H \subseteq C_G(z)$ and D is central, $H \simeq N_K(V_1)/D$ and H is 2-closed iff $N \equiv N_K(V_1)$ is 2-closed. Now it follows from [7, Chapter II, Satz 7.3.a] that $N = R_1L$, where R_1 is an S_2-subgroup of K of order 2^n and L is a cyclic normal subgroup of N of order $(q^2-1)/2^{n-1}$. Since V_1 char R_1, then $N_K(R_1) \subseteq N$ and N is 2-closed iff $N = N_K(R_1)$. However, by [4, Th. 4], $N_K(R_1) \simeq R_1 \times L_1$, where L_1 is a cyclic group of order $(q-1)/2$. Hence, N is 2-closed if and only if $(q^2-1)/2^{n-1} = (q-1)/2$ or $q = 2^{n-2}-1$, and we get Case (I) of Proposition 1.

Possibility (iii) of Case (b) must be dealt with. By [7, Chapter II, Satz 10.12], G is a doubly transitive group of permutations of order $(q^3+1)q^3(q^2-1)/d$, where $d = (3, q+1)$, and of degree q^3+1. The subgroup T of G fixing one letter α is of the type $T = QK$, where Q is a normal subgroup of T of order q^3, which is regular on the remaining q^3 letters, and K is a cyclic group of order $(q^2-1)/d$ Since $q \equiv 1 \pmod 4$, we may assume that $V \subseteq K$. Let $C \equiv C_G(V) = V \times D$; then as $|H/C| = 2$, $H = RD$ and H is 2-closed iff $D \subset C_G(t)$, where t is an involution such that $R = \langle V, t\rangle$. It follows by [11, Th. 9.4] that H acts doubly transitively on Δ, the set of letters fixed by V. Thus H_α is a maximal subgroup of H and since V is an S_2-subgroup of both C and C_α, $CH_\alpha \subset H$ and consequently $CH_\alpha = H_\alpha$. As $[H:C] = 2$, $C = H_\alpha$ and $|\Delta| = 2$. Hence $[H:H_\Delta] = 2$ and $C = H_\Delta \subseteq T - Q$. Thus $(|D|, |Q|) = 1$ and since $K \subseteq C$, $K = H \cap T$. However $V, D \subseteq H \cap T \subset RD$, hence $K = V \times D$. Thus H is 2-closed iff $D \subset C_K(t) \equiv W$, hence iff $K = VW$. As be [9, Lemma 6] $|W| = (q+1)/d$, it follows that R is 2-closed iff $(q^2-1)/d = 2^{n-1}(q+1)/2d$. Thus $PSU(3,q)$, $q \equiv 1 \pmod 4$ satisfies the assumptions of Propositions 1 iff $q \neq 1 + 2^{n-2}$ for some $n \geqq 4$, as stated in (II).

We must deal with Case (a), namely with groups having a dihedral S_2-subgroup. By [6], either (i) $G \simeq A_7$ or (ii) $G \simeq PSL(2,q)$, $q > 3$ odd. Also, in the dihedral case, $|R/V| = 2$. If $G \simeq A_7$, then $C_G(V) = V$ and H is a 2-group, contrary to our assumptions. If G is of type (ii), then H is a dihedral group of order $q+\varepsilon$, where $\varepsilon = \pm 1$. Thus H is 2-closed iff H is a 2-group, hence iff $q = 2^n \pm 1$. We get (III), thus concluding the proof of Proposition 1.

3. Nonsolvable $N_G(V)$

We will prove the following:

PROPOSITION 2. *Under the assumptions of the Theorem, suppose that $N_G(V)$ is nonsolvable. Then either $|V| = 1$ and G is isomorphic to a simple Bender group or $|V| = 2$ and $G \simeq J_{11}$. Conversely, all groups mentioned satisfy the assumptions of the proposition.*

PROOF. Suppose that $|V| = 1$. Then, by the maximality of V, G is a simple

TI-group and by [8, Th. 1], G is isomorphic to a simple Bender group. From now on we will assume that $|V| \geqq 2$ and we will prove that $G \simeq J_{11}$.

Let R be an S_2-subgroup of $N_G(V) \equiv H$ (hence of G), and let S be the maximal solvable normal subgroup of H. We will proceed in a series of steps.

1) *V is a normal S_2-subgroup of S and $S = V \times O(S)$.*

PROOF. Let V_1 be an S_2-subgroup of S. Then $V \subseteq V_1$ and $H = SN_H(V_1)$. Since H is nonsolvable, $N_H(V_1)$ is not 2-closed. The maximality of V forces $V_1 = V$. Since V is cyclic, $S = V \times O(S)$.

2) *H/S is a TI-group.*

PROOF. It follows from (1) and the maximality of V.

3) $H = C_G(V)$.

PROOF. $H/C_G(V)$ is an abelian 2-group. Thus $C_G(V)/S$ is a normal subgroup of H/S of index a power of 2. By [8, Ths. 2 and 3], $C_G(V)/S = H/S$.

4) *If R_1 is an S_2-subgroup of G containing V, then $V \subseteq Z(R_1)$.*

PROOF. In view of (3), is suffices to prove that $R_1 \subseteq H$. Let $R_2 = R_1 \cap H$ and let R_3 be an S_2-subgroup of H containing R_2. Since $R_1 \supset V$, also $R_2 \supset V$. But then the maximality of V forces $R_3 = R_1$, as required.

5) *There exists a normal subgroup L of H containing S, such that $|H:L|$ is odd and L/S is isomorphic to a simple Bender group.*

PROOF. It follows from (2) and [8, Th. 2].

6) $Z(R) \supseteq \Omega_1(R)$.

PROOF. Since $Z(R)S/S \lhd N_{L/S}(RS/S)$, it follows by (5) and the structure of simple Bender groups that either $Z(R)S/S = 1$ or $Z(R)S/S = \Omega_1(RS/S)$. Thus, either $Z(R) = V$ or $\Omega_1(R) \subseteq Z(R)$.

Suppose that $Z(R) = V = \langle v \rangle$; then by [5, Corollary 1], the simplicity of G forces the existence of $g \in G$ such that $v^g \in C_R(v)$, $v^g \neq v$. However, then $V^g \subset R$ and by (4), $V^g \subseteq Z(R)$, which implies that $V^g = V$. Thus $g \in N_G(V) = C_G(V)$ and $v^g = v$, a contradiction.

7) *$V = \langle z \rangle$, z an involution.*

PROOF. Let z be the involution in V. By [5, Corollary 1], there exists in R an involution $y = z^g \neq z$. As by (6) $y \in Z(R)$, we may choose $g \in N_G(R)$. But then $V^g \lhd R$ and $V \cap V^g = 1$. Consequently, $V^g \simeq V^gS/S \lhd RS/S$ and since V^g is cyclic, it follows by (5) and the structure of simple Bender groups that $|V^g| = 2$.

8) *G has at most two conjugate classes of involutions.*

PROOF. Since H/V is a non-2-closed TI-group, it follows by [8, Lemma 6] that H/V has a single conjugacy class of involutions. Let y be a fixed involution in $R - V$ (2-rank $G \geqq 2$) and let u be an arbitrary involution in $R - V$. Then there exists $t \in H = C_G(z)$ such that $uV = y^tV$. Hence either $u = y^t$ or $u = y^tz = (yz)^t$. Since by [5] z is not isolated, it follows that also z is conjugate in G to either y^t or $(yz)^t$.

9) *Either $G \simeq J_{11}$ or G has two conjugate classes of involutions.*

PROOF. Suppose that R is abelian. Since $C_G(z)$ is nonsolvable, it follows by [10] that $G \simeq J_{11}$ and this group indeed satisfies the assumptions of Proposition 2.

Now suppose that R is not abelian. There exists an element ω of R of order 4. By (6), ω^2 is a central involution of R. Suppose that ω^2 is conjugate to z in R; then there exists $t \in N_G(R)$ such that $z = (\omega^2)^t = (\omega^t)^2$ and ω^tV is an involution in H/V. Let y be an involution in $R - V$; then by [8, Lemma 6], there exists $k \in H = C_G(z)$ such that $y^kV = \omega^tV$, which is impossible, since y^kV contains no elements of order 4.

10) *G does not have two conjugate classes of involutions.*

PROOF. Suppose that G is a counter-example and let $|\Omega_1(R)| = q$. Then by (6), R contains $q - 1$ central involutions distributed between two conjugate classes of involutions of G, say K_1 and K_2. We may assume without loss of generality that $|R \cap K_1| = 2k$, k is a positive integer. Consequently $|K_1| = |G : N_G(R)|2k/r$ where r is the number of S_2-subgroups of G containing a fixed element x of K_1. In view of (6), $|K_1|$ is odd, hence r is even. However, again by (6), r is the number of S_2-subgroups of $C_G(x)$, which is odd by the Sylow theorem. This contradiction proves (10).

Proposition 2 follows from the opening remarks, (9), and (10).

The Theorem follows from Propositions 1 and 2.

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